Math 409 Practice Final Exam

Name: ____________________

This exam has 7 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (20 pts)

For each of the following questions, circle the correct answer.

(a) $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 1} =$

A. $\frac{3}{2}$

B. ∞

C. $-\infty$

D. 2

(b) Let f and g be differentiable functions on $\mathbb{R}$ such that $f(0) = 5$, $f'(0) = 2$, and $g'(5) = 3$. Then $(g \circ f)'(0)$ is equal to

A. 6

B. 5

C. 3

D. 2

(c) $\lim_{x \to 0+} (1 + 2x)^{1/x} =$

A. 0

B. 1

C. 2

D. e^2

(d) $\lim_{x\to\infty} \frac{\cos x}{x^2} =$

A. 2

B. 1

C. 0

D. ∞

(e) Which of the following functions is not uniformly continuous on $\mathbb{R}$?

A. $f(x) = \frac{1}{x^2+1}$

B. $f(x) = 1 + x^2$

C. $f(x) = \sin x$

D. $f(x) = \sin^2(x)$

Question 2. (24 pts)

In each of the following 8 cases, indicate whether the given statement is true or false. No justification is necessary.

(a) The image of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is either finite or uncountable.

Solution: True.

(b) If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\{|x_n|\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution: True.

(c) If f is a bounded function on $[0, 1]$, then there is an $a \in [0, 1]$ such that $f(a) = \sup_{x\in[0,1]} f(x)$.

Solution: False.

(d) The function $f(x) = x \sin x$ is integrable on $[0, 5]$.

Solution: True.

(e) For every function $f\colon \mathbb{R} \to \mathbb{R}$, the limit of $f(x)$ as $x \to 0$ exists if and only if $\lim_{n\to\infty} f(x_n)$ exists for all sequences $\{x_n\}_{n=1}^{\infty}$ which converge to 0.

Solution: True.

(f) $\{x_n\}_{n=1}^{\infty}$ is a sequence that converges and $\{y_n\}_{n=1}^{\infty}$ is a sequence that does not converge, then the sequence $\{x_n y_n\}_{n=1}^{\infty}$ dose not converge.

Solution: False.

(g) If $f\colon (0,1) \to \mathbb{R}$ is improperly integrable on $(0,1)$, then f^2 is improperly integrable on $(0,1)$.

Solution: False.

(h) Every nonempty subset of $[0,1]$ has a supremum.

Solution: True.

Question 3. (12 pts)

(a) State the completeness axiom for the real numbers.

Solution: Omitted. You can find it in the textbook.

(b) State the Mean Value Theorem

Solution: Omitted. You can find it in the textbook.

(c) State the Intermediate Value Theorem

Solution: Omitted. You can find it in the textbook.

Question 4. (12 pts)

Compute f' for each of the following functions $f\colon \mathbb{R} \to \mathbb{R}$.

(a) $f(x) = e^{x^2}$

Solution: $f'(x) = e^{x^2} \cdot 2x$.

(b) $f(x) = \displaystyle\int_1^x \frac{t}{2+\cos t} dt$

Solution:

$$f'(x) = \frac{x}{2+\cos x}.$$

(c) $f(x) = \displaystyle\int_1^{x^2} e^{t^2}\, dt$

Solution:

$$f'(x) = e^{x^4} \cdot 2x.$$

Question 5. (12 pts)

(a) State what it means for a function $f\colon \mathbb{R} \to \mathbb{R}$ to be differentiable at a point $a \in \mathbb{R}$.

Solution: Omitted. You can find it in the textbook.

(b) Let f be a function on $\mathbb{R}$ for which there exists a function g such that $f(x) = xg(x)$ for all $x \in \mathbb{R}$ and g is continuous at 0. Prove that $f'(0)$ exists and determine its value.

Solution:

$$f'(0) = \lim_{x\to 0} \frac{xg(x) - 0}{x - 0} = \lim_{x\to 0} g(x) = g(0).$$

where the last equality follows from the assumption that g is continuous at 0.

Question 6. (10 pts)

Define the function $f\colon [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 3, & x = 1 \\ 1, & 1 < x \le 2. \end{cases}$$

Prove directly from the definition of integrability that f is integrable on $[0,2]$.

Solution: For $\forall \varepsilon > 0$, choose a partition $P = \{0, x_1, x_2, 2\}$ with $0 < x_1 < 1 < x_2 < 2$ and $x_2 - x_1 < \varepsilon/3$. Then we have

$$U(f,P) = 0 \cdot (x_1 - 0) + 3 \cdot (x_2 - x_1) + 1 \cdot (2 - x_2)$$

$$L(f,P) = 0 \cdot (x_1 - 0) + 0 \cdot (x_2 - x_1) + 1 \cdot (2 - x_2).$$

It follows that

$$U(f,P) - L(f,P) = 3 \cdot (x_2 - x_1) < \varepsilon.$$

This proves that f is integrable on $[0,2]$.

Question 7. (10 pts)

Prove that the function $f(x) = \dfrac{x}{\sqrt{x^6+1}}$ is improperly integrable on $(0, \infty)$.

Solution: It amounts to prove that f is improperly integrable on $(0, 1]$ and improperly integrable on $[1, \infty)$.

On $(0, 1]$, we have

$$0 \leq \frac{x}{\sqrt{x^6+1}} \leq x.$$

By comparison theorem for improper integrals, f is improperly integrable on $(0, 1]$, since $g(x) = x$ is.

On $[1, \infty)$, we have

$$0 \leq \frac{x}{\sqrt{x^6+1}} \leq \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}.$$

By comparison theorem for improper integrals, f is improperly integrable on $[1, \infty)$, since $h(x) = \frac{1}{x^2}$ is.